

Twin Prime Sieve

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Abstract

A sieve is constructed for ordinary twin primes of the form $6m \pm 1$ that are characterized by their twin rank m . It does not suffer from the parity defect. Non-rank numbers are identified and counted using odd primes $p \geq 5$. Twin- and non-ranks make up the set of positive integers. Regularities of non-ranks allow gathering information on them to obtain a Legendre-type formula for the number of twin-ranks at primordial arguments.

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1 Introduction

Our knowledge of twin primes comes mostly from sieve methods [1],[2],[3],[4]. Conventional sieves, however sophisticated, suffer from the so-called parity defect: In the second member of a pair they cannot distinguish between a prime or product of two primes. The first of many real improvements of Eratosthenes' sieve was achieved by V. Brun [5]. The best result for ordinary twin primes is due to Chen [1],[6] proving there are infinitely many primes p with $p + 2$ either prime or a product of two primes.

Prime numbers $p \geq 5$ are well known to be of the form [7] $6m \pm 1$. An ordinary twin prime occurs when both $6m \pm 1$ are prime.

This paper is based on the original version of Ref. [8]. Our goal here is to develop its mathematical foundations including sieve aspects and asymptotics

for the twin prime counting function from the inclusion-exclusion principle applied to non-ranks (except for the remainder that is estimated in Ref. [9]).

Definition 1.1. If $6m \pm 1$ is an ordinary twin prime pair for some positive integer m , then m is its *twin rank* and $6m$ its *twin index*. A positive integer n is a *non-rank* if $6n \pm 1$ are not both prime.

Since 2, 3 are not of the form $6m \pm 1$ they are excluded as primes in the following.

Example 1. Twin ranks are 1, 2, 3, 5, 7, 10, 12, 17, 18, Twin indices are 6, 12, 18, 30, 42, 60, 72, 102, 108, Non-ranks are 4, 6, 8, 9, 11, 13, 14, 15, 16, 19,

In matters concerning ordinary twin primes, the natural numbers consist of twin- and non-ranks. Only non-ranks have sufficient regularity and abundance allowing us to gather enough information on them to draw inferences on the number of twin-ranks. Therefore, our main focus is on non-ranks, their symmetries and abundance.

In Sect. 2 the twin-prime sieve is constructed based on non-ranks. In Sect. 3 non-ranks are identified in terms of their main properties and then, in Sect. 4, they are counted. In Sect. 5 twin ranks are isolated and then counted. Conclusions are summarized and discussed in Sect. 6.

2 Twin Ranks, Non-Ranks and Sieve

It is our goal here to construct a twin prime sieve. To this end, we need the following arithmetical function.

Definition 2.1. Let x be real. Then $N(x)$ is the integer nearest to x . The ambiguity for $x = n + \frac{1}{2}$ with integral n will not arise in the following.

Lemma 2.2. Let $p \geq 5$ be prime. Then

$$N\left(\frac{p}{6}\right) = \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod{6}; \\ \frac{p+1}{6}, & \text{if } p \equiv -1 \pmod{6}. \end{cases} \quad (1)$$

Proof. This is obvious from Def. 2.1 by substituting $p = 6m \pm 1$. \diamond

Corollary 2.3. If $p \equiv 1 \pmod{6}$ is prime and $p - 2$ is prime, then $\frac{p-1}{6}$ is a twin rank. If $p \equiv -1 \pmod{6}$ and $p + 2$ is prime, then $\frac{p+1}{6}$ is a twin rank.

Proof. This is immediate from Def. 1.1. \diamond

Example 2. This is the case for $p = 7, 13, 19, 31, 43, 61, 73, \dots$ as well as for $p = 5, 11, 17, 29, 41, 59, 71, \dots$ but not for $p = 23, 37, 47, 53, 67, \dots$

Lemma 2.5 *Let $p \geq 5$ be prime. Then all natural numbers*

$$k(n, p)^\pm = np \pm N\left(\frac{p}{6}\right) > 0, \quad n = 1, 2, \dots \quad (2)$$

are non-ranks; there are $2 = 2^{\nu(p)}$ (single) non-rank progressions to p .

(a) If $p \equiv 1 \pmod{6}$ the non-rank $k(n, p)^+$ has

$$6k(n, p)^+ = 6np + (p - 1) \quad (3)$$

sandwiched by the pair

$$([6n + 1]p - 2, [6n + 1]p), \quad (4)$$

and the non-rank $k(n, p)^-$ has

$$6k(n, p)^- = 6np - (p - 1) \quad (5)$$

sandwiched by the pair

$$([6n - 1]p, [6n - 1]p + 2). \quad (6)$$

(b) If $p \equiv -1 \pmod{6}$ the non-rank $k(n, p)^+$ has $6k(n, p)^+ = 6np + (p + 1)$ sandwiched by

$$([6n + 1]p, [6n + 1]p + 2); \quad (7)$$

and the non-rank $k(n, p)^-$ has

$$6k(n, p)^- = 6np - (p + 1) \quad (8)$$

sandwiched by the pair

$$([6n - 1]p - 2, [6n - 1]p). \quad (9)$$

Clearly, these non-ranks are symmetrically distributed at equal distances $N(p/6)$ from multiples of each prime $p \geq 5$. The cases for $n = 0$ are the subject of Cor. 2.3 and Example 1. When there are more than two such non-rank progressions then another prime number will be involved. This issue will be addressed in Sect. 3.

Proof. Let $p \equiv 1 \pmod{6}$ be prime and $n > 0$ an integer. Then $k(n, p)^\pm = np \pm \frac{p-1}{6}$ by Lemma 2.2 and $6np \pm (p - 1)$ are sandwiched by the

two pairs in Eqs. (4),(5) both of which contain a composite number. Hence $k(n, p)^\pm$ are non-ranks.

If $p \equiv -1 \pmod{6}$ and prime, then $k(n, p)^\pm = np \pm \frac{p+1}{6}$ by Lemma 2.2 and $6np \pm (p+1)$ lead to the two pairs in Eqs. (7),(8) both of which contain a composite number again. Hence $k(n, p)^\pm$ are non-ranks. \diamond

The converse of Lemma 2.5 holds, i.e. **non-ranks are prime number driven**.

Lemma 2.6. *If $k > 0$ is a non-rank, there is a prime $p \geq 5$ and a non-negative integer κ so that $k = k(\kappa, p)^+$ or $k = k(\kappa, p)^-$.*

Proof. Let $6k+1$ be composite. Then $6k+1 \neq 2^\mu 3^\nu$, $\mu, \nu \geq 1$ because then $6k+1 \equiv 0 \pmod{6}$, q.e.a. If $\mu = 0$ then $6k+1 \equiv 3 \pmod{6}$, q.e.a. If $\nu = 0$ then $6k+1 \equiv \pm 2 \pmod{6}$, q.e.a. If $6k+1 = 2^\lambda 3^\mu 5^\nu$ then $\lambda + \mu \leq 1$, so $6k+1 = 2^\lambda 5^\nu$ and $6k+1 \equiv (\pm 2)(\pm 1) \pmod{6}$, q.e.a. or $6k+1 = 3^\mu 5^\nu \equiv \pm 3 \pmod{6}$, q.e.a. Hence $6k+1 = p \cdot K$, where $p \geq 5$ is the smallest prime divisor. If $p = 6m+1$, then $K = 6\kappa+1$ and

$$6k+1 = 6^2 m\kappa + 6(m+\kappa) + 1, \quad k = 6m\kappa + m + \kappa = p\kappa + \frac{p-1}{6}. \quad (10)$$

q.e.d. If $p = 6m-1$, then $K = 6\kappa-1$ and

$$6k+1 = 6^2 m\kappa - 6(m+\kappa) + 1, \quad k = 6m\kappa - m - \kappa = p\kappa - \frac{p+1}{6}. \quad (11)$$

The case where $6k-1$ is composite is handled similarly. \diamond

The $k(n, p)^\pm$ yield pairs $6k(n, p)^\pm \pm 1$ with one or two composite entries that are twin-prime analogs of multiples np , $n > 1$ of a prime p in Eratosthenes' prime sieve [7].

Concrete steps to construct a genuine prime pair sieve will be taken in the next section. But it is worth pointing out that many of the non-ranks determined using Lemma 2.5 can be found with the help of primes lower than p , but none of them can be found using primes greater than p . This feature is important because it ensures that, once a number is shown to be a twin rank by some prime up to a certain prime, it is not going to be shown to be a non-rank by any larger primes.

3 Identifying Non-Ranks

Here it is our goal to systematically characterize and identify non-ranks among natural numbers.

Definition 3.1 Let $K > 0$ be integral, $\mathcal{P}_K = \{p : \text{prime}, k(n, p)^+ = K = k(n', p)^-\}$ and $p(K) = \min_{p \in \mathcal{P}_K}$. Then $p(K)$ is called *parent prime* of K .

Example 3.

$$p = 5 : k^+ = 6, 11, 16, 21, \dots; k^- = 4, 9, 14, 19, \dots \quad (12)$$

These k^\pm form the set $\mathcal{A}_5^- = \{5n \pm 1 > 0 : n > 0\} = \mathcal{A}_5$ of non-ranks of parent prime $p = 5$. The initial non-ranks $\mathcal{A}_5^{(0)} = \{4, 6\}$ give all others for $n \geq 1$. Note that 5 is the most effective non-rank generating prime number. If it were excluded like 3 then many numbers, such as 4, 21, 56, 59, 61, 66, 74, 81, 84, 91, 94, \dots , would be missed as non-ranks.

Also $(n+1)p - \frac{p-1}{6} - np - \frac{p-1}{6} = p - \frac{p-1}{3}$ are gaps between non-ranks of the prime $p \equiv 1 \pmod{6}$. So are $np + \frac{p-1}{6} - (np - \frac{p-1}{6}) = \frac{p-1}{3}$.

For $p \equiv -1 \pmod{6}$ the gaps are $(n+1)p - \frac{p+1}{6} - np - \frac{p+1}{6} = p - \frac{p+1}{3}$. And so are $\frac{p+1}{3}$.

Thus, for $p = 5$ the gaps 2, 3 in the set \mathcal{A}_5^- of non-ranks simply alternate.

Proposition 3.3. *The arithmetic progressions $6 \cdot 5n \pm 1, 6(5n+2) \pm 1, 6(5n+3) \pm 1$, $n \geq 0$ contain all twin prime pairs except for 3, 5; 5, 7.*

Note that the arithmetic progression $6(5n+1) + 1$ contains 7 of the twin 5, 7 for $n = 0$ and infinitely many non-twin primes (by Dirichlet's theorem) like 37, while $6(5n+1) - 1$ is composite except for $n = 0$. The set of constants $c \in \mathcal{C}_5 = \{0, 2, 3\}$ of $5n + c$ in Prop. 3.3.

Prop. 3.3 is the first step of the twin-prime sieve.

Proof. From $\{6m \pm 1 : m = 5n, 5n \pm 1, 5n \pm 2, m > 0\}$ we strike all pairs $6(5n+1) \pm 1, 6(5n-1) \pm 1$ resulting from non-ranks of \mathcal{A}_5^- . \diamond

For $p = 7$, we now subtract from the set $\mathcal{A}_7^+ = \{7n \pm 1 > 0 : n > 0\}$ of non-ranks the non-ranks of $p = 5$. The remaining set \mathcal{A}_7 comprises the non-ranks to parent prime $p = 7$.

Lemma 3.4. *The set \mathcal{A}_7 of non-ranks to parent prime $p = 7$ comprises the arithmetic progressions $\{7(5n+1)+1, 7(5n+2) \pm 1, 7(5n+3) \pm 1, 7(5n+4) - 1 : n = 0, 1, 2, \dots\}$.*

The initial non-ranks are $\mathcal{A}_7^{(0)} = \{8, 13, 15, 20, 22, 27\}$.

Proof. The arithmetic progressions $5 \cdot 7n \pm 1 \in \mathcal{A}_5^-, \mathcal{A}_7^+$; so are $5(7n+1)+1 = 7(5n+1)-1 \in \mathcal{A}_5^-, \mathcal{A}_7^+$, $5[7(n+1)-1]-1 = 7(5n+4)+1 \in \mathcal{A}_5^-, \mathcal{A}_7^+$. Subtracting them from \mathcal{A}_7^+ , the arithmetic progressions listed in Lemma 3.4 are left over. \diamond

Note that these four arithmetic progressions contain all common (double) non-ranks of the primes 5, 7.

Proposition 3.5. *The arithmetic progressions $6 \cdot 35n \pm 1, 6(35n + 2) \pm 1, 6(35n + 3) \pm 1, 6(35n + 5) \pm 1, 6(35n + 7) \pm 1, 6(35n + 10) \pm 1, 6(35n + 12) \pm 1, 6(35n + 17) \pm 1, 6(35n + 18) \pm 1, 6(35n + 23) \pm 1, 6(35n + 25) \pm 1, 6(35n + 28) \pm 1, 6(35n + 30) \pm 1, 6(35n + 32) \pm 1, 6(35n + 33) \pm 1, n \geq 0$ contain all twin pairs except for 3, 5; 5, 7.*

In **short notation** we list the constants c of the arithmetic progressions $35n + c$ as $\mathcal{C}_7 = \{c\} = \{0, 2, 3, 5, 7, 10, 12, 17, 18, 23, 25, 28, 30, 32, 33\}$; $\mathcal{C}_5 \subset \mathcal{C}_7$. Except for 0, 28 all c are twin ranks. We call the non-rank 28 to prime 13 an **intruder**.

Proof. Using Lemma 3.4, we strike from the arithmetic progressions of Prop. 3.3 (replacing $n \rightarrow 7n, 7n + 1, \dots, 7n + 6$) all pairs resulting from non-ranks in \mathcal{A}_7 , which are $6[7(5n + 1) + 1] - 1, 6[7(5n + 2) \pm 1] - 1, 6[7(5n + 3) \pm 1] \mp 1, 6[7(5n + 4) - 1] + 1$. This leaves the progressions listed above. The progressions $6(35n + 4) \pm 1, 6(35n + 6) \pm 1$ are missing because each of their pairs contains numbers divisible by 5. \diamond

This is the second step of the sieve.

From Lemma 2.5, for $p = 11$ the set of non-ranks $\mathcal{A}_{11}^- = \{11n \pm 2 : n = 1, 2, \dots\}$. First we subtract from \mathcal{A}_{11}^- the non-ranks of \mathcal{A}_5^- . Common non-ranks of \mathcal{A}_5^- and \mathcal{A}_{11}^- are the arithmetic progressions $5(11n + 2) - 1 = 11(5n + 1) - 2, 5(11n + 5) - 1 = 11(5n + 2) + 2, 5(11n + 6) + 1 = 11(5n + 3) - 2, 5(11n + 9) + 1 = 11(5n + 4) + 2$. Note that again there are four common or double non-rank progressions to the pair of primes 5, 11. Subtracting them from \mathcal{A}_{11}^- this yields the set $\mathcal{A}'_{11} = \{5 \cdot 11n \pm 2, 11(5n + 1) + 2, 11(5n + 2) - 2, 11(5n + 3) + 2, 11(5n + 4) - 2 : n \geq 0\}$ of arithmetic progressions. Next we subtract from \mathcal{A}'_{11} the common (double) non-ranks of \mathcal{A}_7 . Again there are four arithmetic progressions

$$\begin{aligned} 7(11n + 2) - 1 &= 11(7n + 1) + 2, \quad n \geq 0; \\ 7(11n + 3) - 1 &= 11(7n + 2) - 2, \quad n \geq 0; \\ 7(11n + 8) + 1 &= 11(7n + 5) + 2, \quad n \geq 0; \\ 7(11n + 9) + 1 &= 11(7n + 6) - 2, \quad n \geq 0. \end{aligned}$$

This yields \mathcal{A}_{11} , the non-ranks to parent prime $p = 11$. In the short notation of Prop. 3.5, the arithmetic progressions $2 \cdot 3(5 \cdot 7 \cdot 11n + c) \pm 1$ containing all twin primes are $\mathcal{C}_{11} = \{0, 2, 3, 5, 7, 10, 12, 17, 18, 23, 25, 28, 30, 32, 33, 37, 38, 40, 45, 47, 52, 58, 60, 63, 65, 67, 70, 72, 73, 77, 80, 82, 87, 88, 93, 95, 98, 100, 102, 103,$

105, 107, 110, 115, 117, 122, 128, 133, 135, 137, 138, 140, 142, 143, 147, 150, 157, 158, 165, 168, 170, 172, 173, 175, 177, 180, 182, 187, 192, 193, 198, 203, 205, 208, 210, 212, 213, 215, 217, 220, 227, 228, 235, 238, 242, 243, 245, 247, 248, 250, 252, 257, 263, 268, 270, 275, 278, 280, 282, 283, 285, 287, 290, 292, 297, 298, 303, 305, 308, 312, 313, 315, 318, 320, 322, 325, 327, 333, 338, 340, 345, 347, 348, 352, 353, 355, 357, 360, 362, 367, 368, 373, 375, 378, 380, 382} except for 3, 5; 5, 7. Here $28_{13}, 37_{13}, 60_{19}, 63_{13}, 65_{17}, 67_{13}, 73_{19}, \dots$ are intruder non-ranks to the prime listed as subindex, while all other c are twin ranks, except for 0. Note that $\mathcal{C}_7 \subset \mathcal{C}_{11}$, but this pattern does not continue. This completes the 3rd step of the sieve.

We now display characteristic properties of ordinary twin primes that shed light on the pivotal role of $N(p/6)$ and the relevance of non-ranks of Lemmas 2.5 and 2.6 for twin primes.

Theorem 3.6. *Let $p' > p \geq 5$ be primes such that $N(\frac{p'}{6}) = N(\frac{p}{6})$. Then $p' = p + 2$.*

Proof. If $p' \equiv -1 \pmod{6}$ then $N(\frac{p'}{6}) = \frac{p'+1}{6}$. Suppose $p \equiv 1 \pmod{6}$, then $N(\frac{p}{6}) = \frac{p-1}{6}$ and $\frac{p'+1}{6} = \frac{p-1}{6}$. Hence $p' = p - 2$, q.e.a. So $p \equiv -1 \pmod{6}$ and $N(\frac{p}{6}) = \frac{p+1}{6}$ implies $\frac{p'+1}{6} = \frac{p+1}{6}$. Hence $p' = p$, q.e.a. So $p' \equiv 1 \pmod{6}$ and $N(\frac{p'}{6}) = \frac{p'-1}{6}$. Suppose $p \equiv 1 \pmod{6}$, then $N(\frac{p}{6}) = \frac{p-1}{6}$. Since $p' > p$, $N(\frac{p'}{6}) > N(\frac{p}{6})$, q.e.a. Hence $p \equiv -1 \pmod{6}$ and $N(\frac{p}{6}) = \frac{p+1}{6}$. Therefore $\frac{p'-1}{6} = \frac{p+1}{6}$ and $p' = p + 2$. \diamond

Corollary 3.7. *Let $p \geq 5$, $p' = p + 2$ be prime. Then $pp'n \pm \frac{p+1}{6} > 0$ for $n = 0, 1, 2, \dots$ and*

$$\begin{aligned} p(p'n + \frac{p+1}{6}) + \frac{p+1}{6} &= p'(pn + \frac{p+1}{6}) - \frac{p'-1}{6} > 0, \\ p(p'n - \frac{p+1}{6}) - \frac{p+1}{6} &= p'(pn - \frac{p+1}{6}) + \frac{p'-1}{6} > 0, \\ n &= 0, 1, 2, \dots \end{aligned} \tag{13}$$

are their common non-ranks.

Note that, again, there are four arithmetic progressions of common or double non-ranks.

Proof. Using $N(p'/6) = N(p/6) = \frac{p+1}{6}$, Eq. (13) is readily verified; its lhs $\in \mathcal{A}_p^-$ and rhs $\in \mathcal{A}_{p+2}^+$ and $p(p+2)n \pm \frac{p+1}{6} \in \mathcal{A}_p^-, \mathcal{A}_{p+2}^+$. \diamond

Example 4. For $p' = 7$, $p = 5$ the first two common non-ranks in the proof of Lemma 3.4 are cases of Cor. 3.7. For $n = 0$ small non-ranks are obtained.

We now consider more systematically common non-ranks of pairs of primes, generalizing Cor. 3.7 to arbitrary prime pairs p, p' .

Theorem 3.9. *Let $p' > p \geq 5$ be primes. (i) If $p' \equiv p \equiv -1 \pmod{6}$, then $p' = p + 6l$, $l \geq 1$, $N(\frac{p'}{6}) = N(\frac{p}{6}) + l$ and common non-ranks of p', p are, for $n = 0, 1, \dots$,*

$$p(p'n + r') \pm N(\frac{p}{6}) = p'(pn + r) \pm N(\frac{p'}{6}) \quad (14)$$

provided the nonnegative integers r, r' solve

$$(r' - r)p = l(6r \pm 1). \quad (15)$$

Eq. (15) with $6r \pm 1 \equiv 0 \pmod{p}$ on the rhs has a unique solution r that then determines r' .

If r, r' solve

$$(r' - r)p = l(6r \mp 1)l \mp N(\frac{p+1}{6}) \quad (16)$$

then the common non-ranks are

$$p(p'n + r') \pm N(\frac{p}{6}) = p'(pn + r) \mp N(\frac{p'}{6}). \quad (17)$$

(ii) If $p' \equiv p \equiv 1 \pmod{6}$, then $p' = p + 6l$, $l \geq 1$, $N(\frac{p'}{6}) = N(\frac{p}{6}) + l$, and common non-ranks of p', p are

$$p(p'n + r') \pm N(\frac{p}{6}) = p'(pn + r) \pm N(\frac{p'}{6}) \quad (18)$$

provided r, r' solve

$$(r' - r)p = l(6r \pm 1). \quad (19)$$

If r, r' solve

$$(r' - r)p = l(6r \mp 1)l \mp N(\frac{p-1}{6}) \quad (20)$$

then the common non-ranks are

$$p(p'n + r') \pm N(\frac{p}{6}) = p'(pn + r) \mp N(\frac{p'}{6}). \quad (21)$$

(iii) If $p' \equiv 1 \pmod{6}$, $p \equiv -1 \pmod{6}$ then $p' = p + 6l + 2$, $l \geq 0$, $N(\frac{p'}{6}) = N(\frac{p}{6}) + l$, and common non-ranks of p', p are

$$p(p'n + r') \pm N(\frac{p}{6}) = p'(pn + r) \pm N(\frac{p'}{6}) \quad (22)$$

provided

$$(r' - r)p = 6lr + 2r \pm l. \quad (23)$$

If $l = 0$ then $r' = r = 0$ and Eq. (13) are solutions (Cor. 3.7).

If r, r' solve

$$(r' - r)p = 2r(3l + 1)l \mp \left(l + \frac{p+1}{3}\right), \quad l \geq 1, \quad (24)$$

then the common non-ranks are

$$p(p'n + r') \pm N(\frac{p}{6}) = p'(pn + r) \mp N(\frac{p'}{6}). \quad (25)$$

(iv) If $p' \equiv -1 \pmod{6}$, $p \equiv 1 \pmod{6}$ then $p' = p + 6l - 2$, $l \geq 1$, $N(\frac{p'}{6}) = N(\frac{p}{6}) + l$, and common non-ranks of p', p are

$$p(p'n + r') \pm N(\frac{p}{6}) = p'(pn + r) \pm N(\frac{p'}{6}) \quad (26)$$

provided

$$(r' - r)p = 6lr - 2r \pm l. \quad (27)$$

If r, r' solve

$$(r' - r)p = 2r(3l - 1) \mp \left(l + \frac{p-1}{3}\right) \quad (28)$$

then the common non-ranks are

$$p(p'n + r') \pm N(\frac{p}{6}) = p'(pn + r) \mp N(\frac{p'}{6}). \quad (29)$$

Note that, again, there are $4 = 2^{\nu(p'p)}$ arithmetic progressions of common or double non-ranks to the primes p', p in all cases. When there are more than

four non-rank progressions then a 3rd prime will be involved. This case is the subject of Theor. 3.11 below.

Proof. By substituting $p', N(p'/6)$ in terms of $p, N(p/6)$ and l , respectively, it is readily verified that Eqs. (14), (15) are equivalent, as are (16), (17), and (18), (19), and (20), (21), and (22), (23), and (24), (25), and (26), (27), and (28), (29). As in (i) there is a unique solution (r, r') in all other cases as well. \diamond

Example 5. For $p = 5, p' = 7$ we have $l = 0$ and Eq. (23) then gives $5(r' - r) = 2r$, i.e. $r' = r = 0$. Eq. (22) now gives the common non-ranks $35n \pm 1 \in \mathcal{A}_5^-, \mathcal{A}_7^+$. The common non-ranks $5(7n + 6) - 1 = 7(5n + 4) + 1$ are the solution of Eq. (13) with $n \rightarrow n + 1$.

For $p = 5, p' = 11$ we have $l = 1$. Eq. (15) $5(r' - r) = 6r - 1$ gives the solution $r = 1$ and $r' = r + 1 = 2$. Eq. (14) now gives the common non-ranks $5(11n + 2) - 1 = 11(5n + 1) - 2$. Also Eq. (15), $5(r' - r) = 6r + 1$, has the solutions $r = 4, r' = 9$, and Eq. (14) with plus signs displays common non-ranks of 5, 11, $n \geq 0$.

For $p' = 11, p = 7$ Eq. (27), $7(r' - r) = 4r - 1$, has the solutions $r = 2, r' = 3$ and Eq. (26) with minus signs displays common non-ranks of 7, 11 for $n = 0, 1, 2, \dots$

For $p' = 13, p = 5$ Eq. (23), $5(r' - r) = 8r - 1$, has the solutions $r = 2, r' = 5$, and Eq. (22) with minus signs displays common non-ranks of 5, 13, $n \geq 0$.

For $p' = 13, p = 7$ Eq. (19), $7(r' - r) = 6r + 1$, has the solutions $r = 1, r' = 2$, and Eq. (18) with plus signs displays common non-ranks of 7, 13, $n \geq 0$; and Eq. (19), $7(r' - r) = 6r - 1$, has the solutions $r = 6, r' = 11$, so Eq. (18) with minus signs displays common non-ranks of 7, 13, $n \geq 0$.

Theorem 3.11. (Triple non-ranks) *Let $5 \leq p < p' < p''$ (or $5 \leq p < p'' < p'$, or $5 \leq p'' < p < p'$) be different odd primes. Then each case in Theor. 3.9 of four double non-ranks leads to $8 = 2^{\nu(pp'p'')}$ triple non-ranks of p, p', p'' . At two non-ranks per prime, there are at most 2^3 triple non-ranks.*

Proof. It is based on Theor. 3.9 and similar for all its cases. Let's take (i) and substitute $n \rightarrow p''n + \nu$, $0 \leq \nu < p''$ in Eq. (14) which, upon dropping the term $p''p'pn$, yields on the lhs

$$pp'\nu + pr' - N\left(\frac{p}{6}\right) = p''\mu \pm N\left(\frac{p''}{6}\right). \quad (30)$$

Since $(pp', p'') = 1$ there is a unique residue ν modulo p'' so that the lhs of Eq. (30) is $\equiv \pm N\left(\frac{p''}{6}\right) \pmod{p''}$, and this determines μ . As each sign case

leads to such a triple non-rank solution, it is clear that there are 2^3 non-ranks to p, p', p'' . \diamond

Example 6. For 5, 7, 11 the 2^3 triple non-rank progressions are the following. Starting from the double non-rank equations

$$5(11n + 2) - 1 = 11(5n + 1) - 2, \quad (31)$$

replace $n \rightarrow 7n + \nu$, drop $5 \cdot 7 \cdot 11n$ and set the rhs to $7\mu + 1$:

$$5 \cdot 11\nu + 5 \cdot 2 - 1 = 7\mu + 1. \quad (32)$$

Since $5 \cdot 11 + 9 = 7(11 - 2) + 1$ the solution is $\nu = 1$, $\mu = 9$. Putting back $5 \cdot 7 \cdot 11n$ we obtain the triple non-rank system

$$\begin{aligned} 5[11(7n + 1) + 2] - 1 &= 7[11(5n + 1) - 2] + 1 = 7[5(11n + 2) - 1] + 1 \\ &= 11[5(7n + 1) + 1] - 2. \end{aligned} \quad (33)$$

Setting the rhs to $7\mu - 1$ yields the second such solution

$$55\nu + 9 = 7\mu - 1, \quad \nu = 3, \quad \mu = 5^2, \quad (34)$$

$$5[11(7n + 3) + 2] - 1 = 7 \cdot 5(11n + 5) - 1 = 11[7(5n + 2) + 2] - 2. \quad (35)$$

$$5 \cdot 27 - 1 = 7 \cdot 19 + 1 = 11 \cdot 12 + 2 \quad (36)$$

leads to

$$5[11(7n + 2) + 5] - 1 = 7[5(11n + 4) - 1] + 1 = 11[5(7n + 2) + 2] + 2; \quad (37)$$

and solving the other case

$$55\nu + 5^2 - 1 = 7\mu - 1, \quad \nu = 4, \quad \mu = 5 \cdot 7 \quad (38)$$

leads to

$$5[11(7n + 4) + 5] - 1 = 7 \cdot 5(11n + 7) - 1 = 11[7(5n + 3) + 1] + 2. \quad (39)$$

$$5 \cdot (11 \cdot 2 + 6) = 7 \cdot 20 + 1 = 11 \cdot 13 - 2 \quad (40)$$

leads to

$$5[11(7n+2)+6]+1=7\cdot 5(11n+4)+1=11[5(7n+2)+3]-2; \quad (41)$$

and solving

$$5\cdot 11\nu+5\cdot 6+1=7\mu-1, \quad \nu=4, \quad \mu=36 \quad (42)$$

leads to

$$\begin{aligned} 5[11(7n+4)+6]+1 &= 7[5(11n+7)+1]-1=11[7(5n+3)+2]-2 \\ &= 11[5(7n+4)+3]-2. \end{aligned} \quad (43)$$

$$(11\cdot 4-2)+1=7\cdot 5\cdot 6+1=11(5\cdot 4-1)+2 \quad (44)$$

leads to

$$5[11(7n+4)-2]+1=7\cdot 5(11n+6)+1=11[5(7n+4)-1]+2; \quad (45)$$

and then solving

$$5[11(\nu+1)-2]+1=7\mu-1, \quad \nu=5, \quad \mu=46 \quad (46)$$

leads to

$$5[11(7n+6)-2]+1=7[5(11n+9)+1]-1=11[5(7n+6)-1]-2. \quad (47)$$

Theorem 3.13. (Multiple non-ranks) *Let $5 \leq p_1 < \dots < p_m$ be m different primes. Then there are 2^m arithmetic progressions of m -fold non-ranks to the primes p_1, \dots, p_m .*

Proof. This is proved by induction on m . Theors. 3.9 and 3.11 are the $m=2, 3$ cases. If Theor. 3.13 is true for m then for any case $5 \leq p_{m+1} < p_1 < \dots < p_m$, or \dots , $5 \leq p_1 < \dots < p_{m+1}$, we substitute in an m -fold non-rank equation $n \rightarrow p_{m+1}n + \nu$ as in the proof of Theor. 3.11, again dropping the $n \prod_1^{m+1} p_i$ term. Then we get

$$\begin{aligned} & p_1(p_2(\dots(p_m\nu+r_m)+\dots+r_2)+N(\frac{p_1}{6})) \\ &= p_{m+1}\mu \pm N(\frac{p_{m+1}}{6}) \end{aligned} \quad (48)$$

with a unique residue $\nu \pmod{p_{m+1}}$ so that the lhs of Eq. (48) becomes $\equiv N(\frac{p_{m+1}}{6}) \pmod{p_{m+1}}$, which then determines μ . In case the lhs of Eq. (48) has $p_1(\dots) - N(p_1/6)$ the argument is the same. This yields an $(m+1)$ -fold non-rank progression since each sign in Eq. (48) gives a solution. Hence there are 2^{m+1} such non-ranks. At two non-ranks per prime there are at most 2^m non-rank progressions. \diamond

Remark 3.14. In the multiple non-rank equations, $n \prod_1^m p_i$ contains the n -dependence, while the arithmetical details r_i are in other additive terms that are independent of n . This is the reason why the counting of non-ranks in the next sections will be independent of these arithmetical details.

4 Counting Non-Ranks

If we subtract for case (i) in Theor. 3.9, say, the four common non-rank progressions corresponding to the solutions $0 \leq r_i \leq r'_i$, $0 < l_i$ arranged as $0 \leq r_1 \leq r_2 \leq r_3 \leq r_4 < p$ for definiteness, this leaves in $\mathcal{A}_p^- = \{p'n \pm \frac{p'+1}{6} : n \geq 0\}$ the following progressions $p'pn \pm \frac{p'+1}{6}, \dots, p'(np+r_1) + \frac{p'+1}{6}, \dots, p'(np+r_2) - \frac{p'+1}{6}, \dots, p'(np+r_3) + \frac{p'+1}{6}, \dots, p'(np+r_4) - \frac{p'+1}{6}, \dots, p'np \pm \frac{p'+1}{6}$.

We summarize this as follows.

Lemma 4.1. *$p' > p \geq 5$ be prime. Removing the common non-ranks of p', p from the set of all non-ranks of p' leaves arithmetic progressions of the form $p'np + l$; $n \geq 0$ where $l > 0$ are given integers.*

Proposition 4.2. *Let $p > 5$ be prime. Then the set of non-ranks to parent prime p , \mathcal{A}_p , is made up of arithmetic progressions $L(p)n + a$, $n \geq 0$ with $L(p) = \prod_{5 \leq p' \leq p} p$, p' prime and $a > 0$ given integers.*

Proof. Let $p = 6m \pm 1$. We start from the set $\mathcal{A}_p^\pm = \{pn \pm N(\frac{p}{6}) > 0 : n = 0, 1, 2, \dots\}$. Removing the non-ranks common to p and 5 by Lemma 4.1 leaves arithmetic progressions of the form $5pn + l$, $n \geq 0$ where $l > 0$ are given integers. Continuing this process to the largest prime $p' < p$ leaves in \mathcal{A}_p arithmetic progressions of the form $L(p)n + a$, $n \geq 0$ with $L(p) = \prod_{5 \leq p' \leq p} p'$ and $a > 0$ a sequence of given integers independent of n . \diamond

The set $\mathcal{A}_p^{(0)} = \{a\}$ of initial non-ranks repeats in \mathcal{A}_p for $n \geq 1$ with increment (or period) $L(p)$.

Proposition 4.3. *Let $p \geq p' \geq 5$ be primes and $G(p)$ the number of non-ranks $a \in \mathcal{A}_p$ over one period $L(p)$ corresponding to arithmetic progressions $L(p)n + a \in \mathcal{A}_p$. Then $G(p) = 2 \prod_{5 \leq p' < p} (p' - 2)$.*

Note that $G(p) < L(p)$ both increase monotonically as $p \rightarrow \infty$.

Proof. In order to determine $G(p)$ we have to eliminate all non-ranks of primes $5 \leq p' < p$ from \mathcal{A}_p . It suffices to treat the non-ranks a for $n = 0$. As in Lemma 3.4 we start by subtracting the fraction $2/5$ from the interval $1 \leq a \leq L(p)$ of length $L(p)$, then $2/7$ for $p' = 7$ and so on for all $p' < p$. The factor of 2 is due to the symmetry of non-ranks around each multiple of p' according to Lemma 2.5. This leaves $p \prod_{5 \leq p' < p} (p' - 2)$ numbers a . The fraction $2/p$ of these are the non-ranks to parent prime p , which proves Prop. 4.3. \diamond

Prop. 4.3 implies that the fraction of non-ranks related to a prime p in the interval occupied by \mathcal{A}_p ,

$$q(p) = \frac{G(p)}{L(p)} = \frac{2}{p} \prod_{5 \leq p' < p} \frac{p' - 2}{p'}, \quad (49)$$

where p' is prime, decreases monotonically as p goes up.

Definition 4.4. Let $p \geq p' \geq 5$ be prime. The supergroup $\mathcal{S}_p = \bigcup_{5 \leq p' \leq p} \mathcal{A}_{p'}$ contains the sets of non-ranks corresponding to arithmetic non-rank progressions $a + L(p')n$ of all $\mathcal{A}_{p'}$, $p' \leq p$.

Thus, each supergroup \mathcal{S}_p contains nested sets of non-ranks related to primes $5 \leq p' \leq p$.

Let us now count prime numbers from $p_1 = 2$ on.

Proposition 4.5. Let $p_j \geq 5$ be the j th prime. (i) Then the number of non-ranks $a \in \mathcal{A}_{p_i}$ corresponding to arithmetic progressions related to a prime $5 \leq p_i < p_j$,

$$G(p_i) = \frac{L(p_j)}{L(p_i)} G(p_j) = \frac{2L(p_j)}{p_i} \prod_{5 \leq p < p_i} \frac{p - 2}{p} = q(p_i) L(p_j), \quad (50)$$

where p is prime, monotonically decreases as p_i goes up. (ii) The number of non-ranks in a supergroup \mathcal{S}_{p_j} over one period $L(p_j)$ is

$$S(p_j) = L(p_j) \sum_{5 \leq p \leq p_j} q(p) = L(p_j) \left(1 - \prod_{5 \leq p \leq p_j} \frac{p - 2}{p} \right). \quad (51)$$

(iii) The fraction of non-ranks of their arithmetic progressions in the (first) interval $[1, L(p_j)]$ occupied by the supergroup \mathcal{S}_{p_j} ,

$$Q(p_j) = \frac{S(p_j)}{L(p_j)} = \sum_{5 \leq p \leq p_j} q(p) = 1 - \prod_{5 \leq p \leq p_j} \frac{p - 2}{p}, \quad (52)$$

increases monotonically as p_j goes up.

Proof. (i) follows from Prop. 4.3 and Eq. (49). (ii) and (iii) are equivalent and are proved by induction as follows, using Def. 4.4 in conjunction with Eq. (49).

From Eq. (49) we get $q_3 = 2/p_3$ which is the case $j = 3$, $p_j = 5$ of Eq. (52). Assuming Eq. (52) for p_j , we add q_{j+1} of Eq. (49) and obtain

$$\begin{aligned} \sum_{i=3}^{j+1} q(p_i) &= 1 - \prod_{i=3}^j \frac{p_i - 2}{p_i} + \frac{2}{p_{j+1}} \prod_{i=3}^j \frac{p_i - 2}{p_i} \\ &= 1 - \prod_{i=3}^{j+1} \frac{p_i - 2}{p_i}. \end{aligned} \quad (53)$$

The extra factor $0 < (p_{j+1} - 2)/p_{j+1} < 1$ shows that $q(p_j), x(p_j)$ in Eq. (55) decrease monotonically as $p_j \rightarrow p_{j+1}$ while $Q(p_j)$ increases as $j \rightarrow \infty$. \diamond

Definition 4.6. Since $L(p) > S(p)$, there is a set \mathcal{R}_p of *remnants* $r \in [1, L(p)]$ such that $r \notin \mathcal{S}_p$.

Lemma 4.7. (i) The number $R(p_j)$ of remnants in a supergroup, \mathcal{S}_{p_j} , is

$$R(p_j) = L(p_j) - S(p_j) = L(p_j)(1 - Q(p_j)) = \prod_{5 \leq p \leq p_j} (p - 2) = \frac{1}{2}G(p_{j+1}). \quad (54)$$

(ii) The fraction of remnants in \mathcal{S}_{p_j} ,

$$x(p_j) = \frac{R(p_j)}{L(p_j)} = 1 - Q(p_j) = \prod_{5 \leq p \leq p_j} \frac{p - 2}{p}, \quad (55)$$

where p is prime, decreases monotonically as $p_j \rightarrow \infty$.

Proof. (i) follows from Def. 4.6 in conjunction with Eq. (51) and (ii) from Eq. (54). Eq. (54) follows from Eq. (52). \diamond

5 Remnants and Twin Ranks

When all primes $5 \leq p \leq p_j$ and appropriate nonnegative integers n are used in Lemma 2.5 one will find all non-ranks $k < M(j+1) \equiv (p_{j+1}^2 - 1)/6$. By subtracting these non-ranks from the set of positive integers $N \leq M(j+1)$ all and only twin ranks $t < M(j+1)$ are left among the remnants, i.e. twin primes with index $T < p_{j+1}^2 - 1$. If a non-rank k is left then $6k \pm 1$ must have prime divisors that are $> p_j$ according to Lemma 2.5, which is impossible.

Definition 5.1. All $t < M(j+1) = (p_{j+1}^2 - 1)/6$ in a remnant \mathcal{R}_{p_j} of a supergroup \mathcal{S}_{p_j} are twin ranks. These twin ranks are called *front twin ranks*. They are included in R_0 .

Example 7. For $p_{18} = 61$, $p_{19} = 67$ and $M(19) = 748$ we get the set of remnants $m = 1, 2, 3, 5, 7, 10, 12, 17, 18, 23, 25, 30, 32, 33, 38, 40, 45, 47, 52, 58, 70, 72, 77, 87, 95, 100, 103, 107, 110, 135, 137, 138, 143, 147, 170, 172, 175, 177, 182, 192, 205, 213, 215, 217, 220, 238, 242, 247, 248, 268, 270, 278, 283, 287, 298, 312, 313, 322, 325, 333, 338, 347, 348, 352, 355, 357, 373, 378, 385, 390, 397, 425, 432, 443, 448, 452, 455, 465, 467, 495, 500, 520, 528, 542, 543, 550, 555, 560, 562, 565, 577, 578, 588, 590, 593, 597, 612, 628, 637, 642, 653, 655, 667, 670, 675, 682, 688, 693, 703, 705, 707, 710, 712, 723, 737, 747$, which are all twin ranks, i.e., $6m \pm 1$ are prime pairs.

Proposition 5.3. Let p_j be the j th prime number and $L(p_j)n + a_i^{(j)}$ be the arithmetic progressions that contain the non-ranks $a_i^{(j)} \in \mathcal{A}_{p_j}$ to parent prime p_j .

(i) Let $6[L(p_j)n + c_i^{(j)}] \pm 1$ be the arithmetic progressions that contain the ordinary twin primes with $c_i^{(j)} \in \mathcal{C}_{p_j}$. If $c_i^{(j)}$ is a twin rank or intruder non-rank to a prime $p > p_{j+1}$, then $c_i^{(j)} \in \mathcal{C}_{p_{j+1}}$, if it is a non-rank to p_{j+1} then $c_i^{(j)} \notin \mathcal{C}_{p_{j+1}}$.

(ii) The set of constants $c_i^{(j+1)}$ of arithmetic progressions containing the twin ranks from the next supergroup $\mathcal{S}_{p_{j+1}}$ is

$$\begin{aligned} \mathcal{C}_{p_{j+1}} = & \{6[L(p_j)(p_{j+1}n + l) + c_i^{(j)}] \pm 1\} \\ & - \{6[L(p_j)(p_{j+1}n + l') + a_{i'}^{(j)}] \pm 1\}. \end{aligned} \quad (56)$$

If there are positive integers $0 \leq l, l' < p_{j+1}$, a non-rank $a_{i'}^{(j)} \in \mathcal{A}_{p_j}$ and a constant $c_i^{(j)} \in \mathcal{C}_{p_j}$ satisfying

$$L(p_j)l + c_i^{(j)} = L(p_j)l' + a_{i'}^{(j)}, \quad (57)$$

then

$$L(p_j)l + c_i^{(j)} \notin \mathcal{C}_{p_{j+1}}, \quad (58)$$

else

$$c_{i,l}^{(j+1)} = L(p_j)l + c_i^{(j)} \in \mathcal{C}_{p_{j+1}}. \quad (59)$$

Prop. 5.3 is the inductive step completing the practical sieve construction for ordinary twin primes. Props. 3.3, 3.5 and Lemma 3.4 are the initial steps.

Proof. Replacing $n \rightarrow p_{j+1}n$ in (i) we obtain $6[L(p_j)p_{j+1}n + c_i^{(j)}] \pm 1|_{n=0} \in \mathcal{C}_{p_{j+1}}$, if $c_i^{(j)}$ is a twin rank or non-rank to a prime $p > p_{j+1}$. If it is non-rank to p_{j+1} then $c_i^{(j)} \notin \mathcal{C}_{p_{j+1}}$, which proves (i).

Replacing in (ii) $n \rightarrow p_{j+1}n + l$, $l = 1, 2, \dots, p_{j+1} - 1$ and subtracting the resulting sets from each other, we obtain (ii). \diamond

For $p_3 = 5$, Prop. 5.3 is Prop. 3.3, for $p_4 = 7$ it is Prop. 3.5, for $p_5 = 11$ the sequence of c is listed after Prop. 3.5. Clearly, at the start of the c for $p_4 = 7$ the previous values for $p_3 = 5$ are repeated, and this is also the case for $p_5 = 11$; but this pattern does not continue, as shown in Prop. 5.3.

Example 8. For prime number $p_3 = 5$ the (c, n) are given in Prop. 3.3, viz. $(1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0), (3, 3), \dots$ corresponding to the twin primes $6 \pm 1, 30 \pm 1, 60 \pm 1, 12 \pm 1, 42 \pm 1, 72 \pm 1, 18 \pm 1, 108 \pm 1, \dots$, respectively.

For $p_4 = 7$ the additional (c, n) are given in Prop. 3.5, viz. $(10, 0), (17, 0), (18, 0), (23, 0), (25, 0), (32, 0), (33, 0), \dots$ corresponding to the twin primes $60 \pm 1, 102 \pm 1, 108 \pm 1, 138 \pm 1, 150 \pm 1, 180 \pm 1, 192 \pm 1, 198 \pm 1, \dots$, respectively.

For $p_5 = 11$ the additional (c, n) are given in short notation after Prop. 3.5, viz. $(37, 4), (38, 0), (40, 0), \dots$ corresponding to the twin primes $9462 \pm 1, 228 \pm 1, 240 \pm 1, \dots$, respectively.

For $p_{18} = 61$ the twin ranks are listed in Example 7.

Twin ranks are located among the remnants \mathcal{R}_p for any prime $p \geq 5$. The main goal in this Sect. 5 is to establish the inclusion-exclusion principle for non-ranks and use it to derive the twin prime version of Legendre's formula for $\pi(x) - \pi(\sqrt{x})$ extracted from Eratosthenes' sieve [1],[4]. The prime p_j here plays the role of the variable \sqrt{x} there, and the front twin ranks here correspond to the primes $p < \sqrt{x}$ left over after striking out their multiples there.

Theorem 5.5. *Let R_0 be the number of remnants of the supergroup \mathcal{S}_{p_j} , where p_j is the j th prime number, $M(j+1) = [p_{j+1}^2 - 1]/6$ and $x = L(p_j) - M(j+1)$. Then the number of twin ranks within the remnants of the supergroup \mathcal{S}_{p_j} is given by*

$$\pi_2(6x+1) = R_0 + \sum_{p_j < n \leq x, n|L_j(x)} \mu(n) 2^{\nu(n)} \left[\frac{x}{n} \right] + O(1), \quad (60)$$

where $L_j(x) = \prod_{p_j < n \leq x} p$, and $O(1)$ accounts for the less than perfect cancel-

lation at low values of x of R_0 and the sum in Eq. (60). This cancellation is the subject of Theors. 5.7, 5.8 and proved there, up to a remainder estimated in Ref. [9].

The upper limit in the sum (60) is x because $[x/n] = 0$ for $n > x$. Here $L(p_j) = \prod_{5 \leq p \leq p_j} p$, and $R_0 = \prod_{5 \leq p \leq p_j} (p - 2)$ with p prime includes the front twin ranks, and n runs through all products of primes $p_j < p \leq x$.

The argument of the twin-prime counting function π_2 is $6x+1$ because, if x is the last twin rank of the interval $[1, L(p_j)]$, then $6x \pm 1$ are the corresponding twin primes. The twin pair 3, 5 has no twin rank and is not part of the remnants.

The formula (60) may be regarded as the twin-prime analog of Legendre's application of Erathostenes' sieve to the prime counting function $\pi(x)$ in terms of Möbius' arithmetic function which was subsequently improved by many others [3].

Proof. According to Prop. 4.5 the supergroup \mathcal{S}_{p_j} has $S(p_j) = L(p_j) \cdot \left(1 - \prod_{5 \leq p \leq p_j} \frac{p-2}{p}\right)$ non-ranks. Subtracting these from the interval $[1, L(p_j)]$ that the supergroup occupies gives $R_0 = \prod_{5 \leq p \leq p_j} (p - 2)$ for the number of remnants which include twin ranks and non-ranks to primes $p_j < p \leq x$. The latter are

$$M(j+1) < pn \pm N\left(\frac{p}{6}\right) \leq L(p_j), \quad M(j+1) = \frac{p_{j+1}^2 - 1}{6}, \quad (61)$$

or

$$0 < n \leq \frac{L(p_j) - M(j+1)}{p}, \quad (62)$$

which have to be subtracted from the remnants to leave just twin ranks. Correcting for double counting of common non-ranks to two primes using Theor. 3.9, of triple non-ranks using Theor. 3.11 and multiple non-ranks using Theor. 3.13 we obtain

$$\begin{aligned} \pi_2(6x+1) &= R_0 - 2 \sum_{p_j < p \leq x, n | L_j(x)} \left\lfloor \frac{x}{p} \right\rfloor \\ &+ 4 \sum_{p_j < p < p' \leq x} \left\lfloor \frac{x}{pp'} \right\rfloor \mp \cdots + O(1), \end{aligned} \quad (63)$$

where $[x]$ is the integer part of x as usual and where $L_j(x) = \prod_{p_j < p \leq x} p$. Equation (63) is equivalent to Eq. (60). \diamond

Definition 5.6 Replacing the floor function $[x]$ in Eq. (63) by its argument minus fractional part, $[x] = x - \{x\}$, we call

$$\begin{aligned} R_M &= R_0 + \sum_{p_j < n \leq x, n|L_j(x)} \mu(n) 2^{\nu(n)} \frac{x}{n}, \\ R_E &= - \sum_{p_j < n \leq x, n|L_j(x)} \mu(n) 2^{\nu(n)} \left\{ \frac{x}{n} \right\} + O(1) \end{aligned} \quad (64)$$

the main and error terms of Eq. (60) or Eq. (63).

Theorem 5.7. *The main term R_M in Eq. (60) satisfies*

$$\begin{aligned} R_M &= L(p_j) \prod_{5 \leq p \leq x} \left(1 - \frac{2}{p} \right) \\ &\quad + M(j+1) \left[1 - \prod_{p_j < p \leq x} \left(1 - \frac{2}{p} \right) \right]. \end{aligned} \quad (65)$$

Proof. Expanding the product

$$R_0 = L(p_j) \prod_{5 \leq p \leq p_j} \left(1 - \frac{2}{p} \right) \quad (66)$$

and combining corresponding sums of R_M in Eq. (64)

$$- \sum_{5 \leq p \leq p_j} \frac{1}{p} - \sum_{p_j < p \leq x} \frac{1}{p} = - \sum_{5 \leq p \leq x} \frac{1}{p}, \dots \quad (67)$$

just shifts the upper limit p_j in the product $\prod_{p \leq p_j} (1 - 2/p)$ to x so that we obtain Eq. (65). \diamond

Theorem 5.8. *The main term R_M obeys the asymptotic law*

$$R_M \sim \frac{c_2 e^{-2\gamma} 6x}{\log^2(6x+1)}, \quad (68)$$

as the j th prime $p_j \sim \log x \rightarrow \infty$ where c_2 is the twin prime constant.

Proof. The ratio of the second term in Eq. (65) to the leading first term is of order,

$$\frac{p_{j+1}^2 \log^2 x}{x} \rightarrow 0, \quad p_{j+1} = O(\log x). \quad (69)$$

Using the prime-number theorem [7],[2] we have

$$\log L(p_j) = \sum_{5 \leq p \leq p_j} \log p = p_j + R(p_j) = \log x + O\left(\frac{\log^2 x}{x}\right) \quad (70)$$

for all sufficiently large primes p_j , $R(p_j)$ is the remainder in the prime number theorem and the error term comes from $M(j+1)$.

Using Mertens' asymptotic formula [2] for $x \rightarrow \infty$:

$$\prod_{p=2}^x \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}, \quad \gamma \approx 0.5772, \quad (71)$$

and the twin-prime constant

$$c_2 = \frac{\prod_{p>2} (1 - \frac{2}{p})}{\prod_{p>2} (1 - \frac{1}{p})^2} \quad (72)$$

where p runs through primes, we obtain

$$\begin{aligned} \prod_{p>2}^x \left(1 - \frac{2}{p}\right) &= \frac{\prod_{p>2}^x (1 - \frac{2}{p})}{\prod_{p>2}^x (1 - \frac{1}{p})^2} \prod_{p>2}^x \left(1 - \frac{1}{p}\right)^2 \sim c_2 \prod_{p>2}^x \left(1 - \frac{1}{p}\right)^2 \\ &\sim \frac{4c_2 e^{-2\gamma}}{\log^2 x}, \quad x \rightarrow \infty \end{aligned} \quad (73)$$

and thus

$$R_M \sim \frac{c_2 e^{-2\gamma} 6x}{\log^2(6x+1)}, \quad \log x \rightarrow \infty \quad (74)$$

from Theorem 5.7. \diamond

6 Summary and Discussion

The twin prime sieve constructed here differs from other more general sieves that are applied to twin primes among many other problems in that it is conceptually designed for ordinary prime twins. It is specific rather than general and has no precursor in sieve theory. Hence much of the work is devoted to developing its concepts into sieve tools. Accurate counting of non-rank sets require the infinite primorial set $\{6L(p_j) = \prod_{p \leq p_j} p\}$.

The twin primes are not directly sieved, rather twin ranks m are with $6m \pm 1$ both prime. All other natural numbers are non-ranks. These are much more numerous and orderly than twin ranks. Surprisingly, their order is governed by all primes $p \geq 5$. In contrast to other sieves, primes serve to organize and classify non-ranks in arithmetic progressions with equal distances (periods) that are primes (≥ 5) or products of them.

The coefficient $c_2 e^{-2\gamma} \approx 0.416213$ in the asymptotic law of the main term R_M of $\pi_2(6x + 1)$ in Theor. 5.8 is a little less than a third of the Hardy-Littlewood constant $2c_2 \approx 1.320320$. In Ref. [9] the remainder R_E is shown to be at most of the order of the main term divided by any positive power of $\log x$. To put the deviation from the Hardy-Littlewood law in perspective, our “minimal” asymptotic law holds only near primorials. In the large gaps between those special arguments there is room for other asymptotic laws.

Finally, needless to say, the genuine sieve has no consequences for other twin primes or the Goldbach problem [7].

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